Non-Euclidian Properties of Plane Cubics.

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Take $\Omega_{xx}=0$ as the equation of the absolute; $\Omega_{xy}=0$ the equation of the polar of the point y with respect to the absolute; $C_{xx}=0$ the equation of the cubic. Then $\Omega_{ba}=0$ is the condition that the point b lie on the polar of the point a with respect to the absolute. Take (xy) as the non-Euclidian distance from the point x to the point y, or between the polars of those points with respect to the absolute, $\overline{xy}$, the corresponding distance from the point x to the polar of the point y with respect to the absolute.

Then we have:

$$\cos\frac{(xy)}{2ki}=\sin\frac{\overline{xy}}{2ki}=\frac{\Omega_{xy}}{\sqrt{\Omega_{xx}\,\Omega_{yy}}}.$$

It seems desirable to use, in any discussion of non-Euclidian properties of curves, certain terms used by W. K. Clifford and also by Professor Story in regard to certain properties of the conic, since they apply to any curve, viz. There are six intersections of the cubic with the absolute, the *absolute points*. Twelve common tangents to cubic and absolute, the *absolute tangents*. Six tangents to the cubic at the absolute points, the *asymptotes*. Fifteen lines joining the absolute points in pairs, the *focal lines*. Sixty-six intersections of pairs of absolute tangents, the *foci*. Twelve points of contact of the absolute tangents with the absolute, the *asymptotic points*. The lines joining the asymptotic points are the *directrices*. The intersections of the asymptotes are the *directors*.

Then there are certain other terms which apply to the cubic alone. Related to any pair of tangents there is a third tangent; viz., the tangent to the cubic at the third intersection with the cubic of the chord of contact of the two given tangents. Such third tangents will be spoken of as *third absolute tangents*, etc., depending upon the character of the original pair of tangents. Associated with

three such tangents there is a satellite line, which will be designated as an *absolute satellite line*, etc. Then there are certain first and second polars, with respect to the cubic, of fixed points and lines. These are designated by prefixing the special name of the fixed point or line as *absolute tangent first polar*, etc. Lastly, *N. E. D.* is used to designate the non-Euclidian distance divided by the proper constant, i. e., $\frac{(xy)}{2ki}$.

Consider the six absolute points. Connect them by three straight lines, no point being on more than one of the lines. Then if a, b, c are the directors of these focal lines, their equations are:

$$\Omega_{xa} = 0,$$
$$\Omega_{xb} = 0,$$
$$\Omega_{xc} = 0.$$

Each of these lines cuts the cubic in a third point. These three points, say p_1, p_2, p_3, lie on a line, $C = 0$.

Then the equation of the cubic may be written in the three ways:

$$C_{xx} \equiv \lambda_1 \Omega_{xa} \Omega_{xx} - C\Omega_{xb} \Omega_{xc} = 0,$$
$$C_{xx} \equiv \lambda_2 \Omega_{xb} \Omega_{xx} - C\Omega_{xa} \Omega_{xc} = 0,$$
$$C_{xx} \equiv \lambda_3 \Omega_{xc} \Omega_{xx} - C\Omega_{xa} \Omega_{xb} = 0,$$

or what is the same thing,

$$\frac{\lambda_1}{C} = \frac{\Omega_{xb} \Omega_{xc}}{\Omega_{xa} \Omega_{xx}}, \tag{1}$$

$$\frac{\lambda_2}{C} = \frac{\Omega_{xa} \Omega_{xc}}{\Omega_{xb} \Omega_{xx}}, \tag{2}$$

$$\frac{\lambda_3}{C} = \frac{\Omega_{xa} \Omega_{xb}}{\Omega_{xc} \Omega_{xx}}. \tag{3}$$

But the right-hand member of (1) may be written:

$$\frac{\dfrac{\Omega_{xb}}{\sqrt{\Omega_{xx} \Omega_{bb}}} \cdot \dfrac{\Omega_{xc}}{\sqrt{\Omega_{xx} \Omega_{cc}}} \cdot \sqrt{\dfrac{\Omega_{bb} \Omega_{cc}}{\Omega_{aa} \Omega_{xx}}}}{\dfrac{\Omega_{xa}}{\sqrt{\Omega_{xx} \Omega_{aa}}}}.$$

Therefore, (1) may be written:

$$\frac{\lambda_1 \sqrt{\Omega_{aa}} \sqrt{\Omega_{xx}}}{C \cdot \sqrt{\Omega_{bb} \Omega_{cc}}} = \frac{\cos \dfrac{(xb)}{2ki} \cdot \cos \dfrac{(xc)}{2ki}}{\cos \dfrac{(xa)}{2ki}}, \tag{4}$$

similarly for (2) and (3).

$$\lambda_2 \frac{\sqrt{\Omega_{bb}}\sqrt{\Omega_{xx}}}{C.\sqrt{\Omega_{aa}\Omega_{cc}}} = \frac{\cos\frac{(xa)}{2ki}\cdot\cos\frac{(xc)}{2ki}}{\cos\frac{(xb)}{2ki}}, \tag{5}$$

$$\lambda_3 \frac{\sqrt{\Omega_{cc}}\sqrt{\Omega_{xx}}}{C.\sqrt{\Omega_{aa}\Omega_{bb}}} = \frac{\cos\frac{(xa)}{2ki}\cdot\cos\frac{(xb)}{2ki}}{\cos\frac{(xc)}{2ki}}. \tag{6}$$

Dividing (4) by (5) we have

$$\frac{\lambda_1}{\lambda_2}\cdot\frac{\Omega_{aa}}{\Omega_{bb}} = \frac{\cos^2\frac{(xb)}{2ki}}{\cos^2\frac{(xa)}{2ki}} = \frac{\sin^2\frac{\overline{xb}}{2ki}}{\sin^2\frac{\overline{xa}}{2ki}},$$

and similar relations from (4) and (6), (5) and (6). The left-hand member is constant, hence the theorem:

The ratio of the squares of the cosines (sines) of the N. E. D. from any point of a cubic to any pair of directors whose focal lines do not have a common absolute point (to any pair of focal lines not through the same absolute point) is constant.

If the lines are such that some of them pass through the same absolute point, then the above ratio multiplied by $\frac{C}{C'}$ is constant.

Take T and T' a pair of absolute tangents; H the corresponding third absolute tangent; Q the chord of contact and S the absolute satellite line. Then if the poles of these lines, with respect to the absolute, are represented by the corresponding small letters, we may write:

$$C_{xx} \equiv \Omega_{xt}\,\Omega_{xt'}\,\Omega_{xh} - \lambda\Omega^2_{xq}\,\Omega_{xs} = 0. \tag{7}$$

This gives

$$\frac{\frac{\Omega_{xt}}{\sqrt{\Omega_{xx}\Omega_{tt}}}\cdot\frac{\Omega_{xt'}}{\sqrt{\Omega_{xx}\Omega_{t't'}}}\cdot\frac{\Omega_{xh}}{\sqrt{\Omega_{xx}\Omega_{hh}}}}{\frac{\Omega^2_{xq}}{\Omega_{xx}\Omega_{qq}}\cdot\frac{\Omega_{xs}}{\sqrt{\Omega_{xx}\Omega_{ss}}}} = \lambda\sqrt{\frac{\Omega_{tt}\,\Omega_{t't'}\,\Omega_{hh}}{\Omega^2_{qq}\,\Omega_{ss}}},$$

or

$$\frac{\sin\frac{\overline{xt}}{2ki}\cdot\sin\frac{\overline{xt'}}{2ki}\cdot\sin\frac{\overline{xh}}{2ki}}{\sin^2\frac{xq}{2ki}\cdot\sin\frac{xs}{2ki}} = \text{const.} \tag{8}$$

Hence:

The ratio of the product of the sines of the N. E. D. from any point of a cubic to any pair of absolute tangents and third absolute tangent, to that of the square of the sine of the N. E. D. to the chord of contact into the sine of the N. E. D. to the absolute satellite line, is constant.

For the absolute points we should have equation (7), and also the equation of the absolute, which may be written $TT' - \Omega^2_{xa} = 0$, where a is the corresponding focus, simultaneously true. Hence, for such points, say a point m, we may write:

$$\Omega^2_{ma}\,\Omega_{mb} = \lambda\Omega^2_{mq}\,\Omega_{ms},$$

or

$$\left[\frac{\Omega_{ma}}{\Omega_{mq}}\right]^2 = \frac{\lambda\Omega_{ms}}{\Omega_{mh}},$$

which gives

$$\frac{\sin^2 \frac{\overline{ma}}{2ki}}{\sin^2 \frac{\overline{mq}}{2ki}} = \lambda\,\frac{\sin \frac{\overline{ms}}{2ki}}{\sin \frac{\overline{mh}}{2ki}}$$

as a relation connecting an absolute point m with the other quantities involved in the preceding theorem.

Take a_1, a_2, a_3 the poles, with respect to the absolute, of the tangents at three collinear points of inflexion; c that of their chord of contact.

Then we may write:

$$C_{xx} \equiv \Omega_{xa_1}\,.\,\Omega_{xa_2}\,.\,\Omega_{xa_3} - \lambda\Omega^3_{xc} = 0,$$

or

$$\frac{\frac{\Omega_{xa_1}}{\sqrt{\Omega_{xx}\,\Omega_{a_1a_1}}}\cdot\frac{\Omega_{xa_2}}{\sqrt{\Omega_{xx}\,\Omega_{a_2a_2}}}\cdot\frac{\Omega_{xa_3}}{\sqrt{\Omega_{xx}\,\Omega_{a_3a_3}}}}{\frac{\Omega^3_{xc}}{(\sqrt{\Omega_{xx}\,\Omega_{cc}})^3}} = \lambda\,\frac{\sqrt{\Omega_{a_1a_1}\,\Omega_{a_2a_2}\,\Omega_{a_3a_3}}}{\Omega^{\frac{3}{2}}_{cc}} = \text{const.}$$

This gives

$$\frac{\sin\frac{\overline{xa_1}}{2ki}\cdot\sin\frac{\overline{xa_2}}{2ki}\cdot\sin\frac{\overline{xa_3}}{2ki}}{\sin^3\frac{\overline{xc}}{2ki}} = \text{const.}$$

Hence the theorem:

The ratio of the product of the sines of the N. E. D. from any point of a cubic to the tangents at three collinear points of inflexion, to the cube of the sine of the N. E. D. to the chord of contact is constant.

If $\Omega_{xa}=0$ and $\Omega_{xb}=0$ are two lines cutting the cubic in three sets of points; $\Omega_{xa_1}=0$, $\Omega_{xa_2}=0$ and $\Omega_{xa_3}=0$ be lines joining corresponding pairs of these points; and $\Omega_{xc}=0$ be the line on which the remaining intersections must lie, then we may write:

$$C_{xx}\equiv \Omega_{xa_1}\,\Omega_{xa_2}\,\Omega_{xa_3}-k\Omega_{xa}\,\Omega_{xb}\,\Omega_{xc}=0.$$

Treating this as before, we have

$$\frac{\sin\frac{\overline{xa_1}}{2ki}\cdot\sin\frac{\overline{xa_2}}{2ki}\cdot\sin\frac{\overline{xa_3}}{2ki}}{\sin\frac{\overline{xa}}{2ki}\cdot\sin\frac{\overline{xb}}{2ki}\cdot\sin\frac{\overline{xc}}{2ki}}=k\sqrt{\frac{\Omega_{a_1a_1}\,\Omega_{a_2a_2}\,\Omega_{a_3a_3}}{\Omega_{aa}\,\Omega_{bb}\,\Omega_{cc}}}.\tag{A}$$

Since all the lines are fixed if $\Omega_{xa}=0$ and $\Omega_{xb}=0$, therefore, the left-hand ratio is constant for any two fixed lines $\Omega_{xa}=0$ and $\Omega_{xb}=0$. This relation applies best to focal lines—since, then, two of the chords of contact are also focal lines—as follows:

Consider four focal lines through four absolute points—two through each point. Say Ω_{xa_1}, Ω_{xa_2}, Ω_{xa_3}, Ω_{xa_4}, where a_1, a_2, a_3, a_4 are the corresponding directors; take Ω_{xb} and Ω_{xc} the chords of contact of the third intersections of pairs of focal lines not through a common point. Then from relation (A) we write:

$$\frac{\sin\frac{\overline{xa_1}}{2ki}\cdot\sin\frac{\overline{xa_2}}{2ki}\cdot\sin\frac{\overline{xb}}{2ki}}{\sin\frac{\overline{xa_3}}{2ki}\cdot\sin\frac{\overline{xa_4}}{2ki}\cdot\sin\frac{\overline{xc}}{2ki}}=\text{const.}\tag{B}$$

Hence:

The ratio of the sines (cosines) of the N. E. D. between any point of a cubic and any pair of focal lines, not through the same absolute point, and between the chord of the third intersections of such a pair (to the directors of such a pair of focal lines and to the pole, with respect to the absolute, of the chord of third intersections) to the corresponding product for any other like pair of focal lines, is constant.

If we consider the six possible focal lines through four absolute points, there would be three concurrent lines, $\Omega_{xb}=0$, $\Omega_{xc}=0$, $\Omega_{xd}=0$, where the first two have the same meaning as above and $\Omega_{xd}=0$ is the corresponding sine for the other pair of focal lines. We should find six relations like (B), but only two of them independent, viz. (B) and (C) where (C) equals

$$\frac{\sin\frac{\overline{xa_5}}{2ki}\cdot\sin\frac{\overline{xa_6}}{2ki}\cdot\sin\frac{\overline{xb}}{2ki}}{\sin\frac{\overline{xa_3}}{2ki}\cdot\sin\frac{\overline{xa_4}}{2ki}\cdot\sin\frac{\overline{xd}}{2ki}}=\text{const.}\tag{C}$$

where a_i $(i=1,\ldots.\,6)$ are the directors of the six focal lines.

The product of these two independent relations gives a relation *between the six focal lines through any four absolute points and the three concurrent lines on which their third intersections lie, viz.*

$$\frac{\sin\frac{\overline{xa_1}}{2ki}\cdot\sin\frac{\overline{xa_2}}{2ki}\cdot\sin\frac{\overline{xa_5}}{2ki}\cdot\sin\frac{\overline{xa_6}}{2ki}\cdot\sin^2\frac{\overline{xb}}{2ki}}{\sin^2\frac{\overline{xa_3}}{2ki}\cdot\sin^2\frac{\overline{xa_4}}{2ki}\cdot\sin\frac{\overline{xc}}{2ki}\cdot\sin\frac{\overline{xd}}{2ki}}=\text{const.}\tag{D}$$

Since the chords joining asymptotic points are directrices, it follows that relations corresponding to (B), (C) and (D) hold for directrices, if we replace focal lines of original pair by directrices, directors by foci and absolute points by the intersections of the directrix with the cubic; here, evidently, the chords will not be directrices.

If we consider a pair of asymptotes, we may use equation (8). We have the two tangents as asymptotes, and the chord of contact is a focal line. Then we have the third asymptotic tangent and asymptotic satellite line. We have then at once the relation:

The ratio of the product of the sines of the N. E. D. from any point of a cubic to any pair of asymptotic tangents and third asymptotic tangent to the product of the square of the sine of the N. E. D. to the corresponding focal line into the sine of the N. E. D. to the asymptotic satellite line, is constant.

Consider the triangle with vertices at any two directors and any point of the cubic. Take a and b the lines joining the point of the cubic to the directors; c the line joining the directors; α, β, γ their opposite angles. Then, from the first theorem of this paper we have:

$\dfrac{\cos^2\frac{a}{2ki}}{\cos^2\frac{b}{2ki}}=\lambda$; the relation $\dfrac{\sin^2\frac{a}{2ki}}{\sin^2\frac{b}{2ki}}=\dfrac{\sin^2\frac{\alpha}{2k'}}{\sin^2\frac{\beta}{2k'}}$ is also true, which gives

$$\frac{1-\cos^2\frac{a}{2ki}}{1-\cos^2\frac{b}{2ki}}=\frac{\dfrac{1}{\cos^2\frac{b}{2ki}}-\dfrac{\cos^2\frac{a}{2ki}}{\cos^2\frac{b}{2ki}}}{\dfrac{1}{\cos^2\frac{b}{2ki}}-1}=\frac{\sec^2\frac{b}{2ki}-\lambda}{\sec^2\frac{b}{2ki}-1}=\frac{\sin^2\frac{\alpha}{2k'}}{\sin^2\frac{\beta}{2k'}}$$

This gives
$$\frac{\sin^2\frac{\beta}{2k'} - \sin^2\frac{\alpha}{2k'}\cdot\sin^2\frac{b}{2ki}}{\cos^2\frac{b}{2ki}\cdot\sin^2\frac{\beta}{2k'}} = \text{const.}$$

Similarly, we obtain the corresponding relation involving a, viz.
$$\frac{\sin^2\frac{\alpha}{2k'} - \sin^2\frac{\beta}{2k'}\sin^2\frac{a}{2ki}}{\cos^2\frac{a}{2ki}\sin^2\frac{\alpha}{2k'}} = \text{const.}$$

These constants are reciprocals, hence we have :

If a and b are the non-Euclidian distances from any point of a cubic to any pair of directors and α, β the angles which they make with the line joining the directors, then we have the relation:

$$\begin{vmatrix} \sin^2\frac{\alpha}{2k'} & \sin^2\frac{\beta}{2k'} \\ \sin^2\frac{a}{2ki} & 1 \end{vmatrix} \cdot \begin{vmatrix} \sin^2\frac{\beta}{2k'} & \sin^2\frac{\alpha}{2k'} \\ \sin^2\frac{b}{2ki} & .1 \end{vmatrix} = \cos^2\frac{a}{2ki}\cdot\cos^2\frac{b}{2ki}\cdot\sin^2\frac{\alpha}{2k'}\cdot\sin^2\frac{\beta}{2k'}.$$

We also have the relation:
$$\cos\frac{a}{2ki} = \cos\frac{b}{2ki}\cos\frac{c}{2ki} + \cos\frac{\alpha}{2k'}\sin\frac{a}{2ki}\sin\frac{b}{2ki}\cdot\frac{\cos\frac{a}{2ki}}{\cos\frac{b}{2ki}} \text{ and } \cos\frac{c}{2ki}$$
are constant, hence we may write: $\cos\frac{\alpha}{2k'}\sin\frac{a}{2ki}\tan\frac{b}{2ki} = \text{const.}$, and similarly, $\cos\frac{\beta}{2k'}\sin\frac{b}{2ki}\tan\frac{a}{2ki} = \text{const.}$, which may be written,
$$\begin{cases} \sin\frac{a}{2ki}\sin\frac{b}{2ki}\dfrac{\cos\frac{\alpha}{2k'}}{\cos\frac{b}{2ki}} = \text{const.}, \\ \sin\frac{a}{2ki}\sin\frac{b}{2ki}\dfrac{\cos\frac{\beta}{2k'}}{\cos\frac{a}{2ki}} = \text{const.} \end{cases}$$

Hence, we may say:

The product of the sines of the N. E. D. from any point of a cubic to any pair

of directors multiplied by the ratio of the cosine of the angle at either director to the cosine of the N. E. D. of the adjacent side, is constant.

We may write:

$$\sin\frac{c}{2ki} - \cos\frac{c}{2ki}\tan\frac{b}{2ki}\cos\frac{\alpha}{2k'} = \lambda\sin\frac{\alpha}{2k'}\cos\frac{\beta}{2k'},$$

$$\sin\frac{c}{2ki} - \cos\frac{c}{2ki}\tan\frac{a}{2ki}\cos\frac{\beta}{2k'} = \frac{1}{\lambda}\sin\frac{\beta}{2k'}\cos\frac{\alpha}{2k'},$$

where λ is constant and equal to $\dfrac{\sin\frac{a}{2ki}}{\cos\frac{a}{2ki}}$; calling $\sin\frac{c}{2ki} = k$ and $\cos\frac{c}{2ki} = h$, we have by subtraction:

$$\tan\frac{a}{2ki}\cdot\cos\frac{\beta}{2k'} - \tan\frac{b}{2ki}\cos\frac{\alpha}{2k'}$$
$$= \frac{1}{h}\left[\sin\frac{\alpha}{2k'}\cos\frac{\beta}{2k'} - \frac{1}{\lambda}\sin\frac{\beta}{2k'}\cos\frac{\alpha}{2k'}\right].$$

Hence, we have: *The ratio* $\dfrac{\begin{vmatrix}\tan\frac{a}{2ki} & \cos\frac{\alpha}{2k'}\\ \tan\frac{b}{2ki} & \cos\frac{\beta}{2k'}\end{vmatrix}}{\begin{vmatrix}\lambda\sin\frac{\alpha}{2k'} & \cos\frac{\alpha}{2k'}\\ \frac{1}{\lambda}\sin\frac{\beta}{2k'} & \cos\frac{\beta}{2k'}\end{vmatrix}}$ *is constant for any point of a cubic and any pair of directors.*

Starting with the first theorem of this paper, and taking a and b to be the distances from any point of a cubic to any pair of focal lines not concurrent at the same absolute point; c the distance between their intersections with the pair of focal lines, and α, β, γ the opposite angles. We may write at once:

$$\frac{\sin^2\frac{\alpha}{2k'}}{\sin^2\frac{\beta}{2k'}} = \frac{\sin^2\frac{a}{2ki}}{\sin^2\frac{b}{2ki}} = \text{const.}$$

Hence:

The ratio of the squares of the sines of the angles which the lines from any point

of a cubic to any pair of focal lines, not concurrent at the same absolute point, make with the line joining the intersections, is constant.

In relation (8), join the intersection of $\overline{xt}$ with its absolute tangent to the intersection of $\overline{xy}$ with the chord of contact and call the angles thus formed at the intersections δ and α; connect the last intersection with the intersection or $\overline{xt'}$ with its tangent and call the angles β, δ_1; lastly, join the intersection of $\overline{xh}$ with its tangent to the intersection of $\overline{xs}$ with the satellite and call the angles γ, π. Then we have the relation:

$$\frac{\sin\dfrac{\alpha}{2k'}\cdot\sin\dfrac{\beta}{2k'}\cdot\sin\dfrac{\gamma}{2k'}}{\sin\dfrac{\delta}{2k'}\cdot\sin\dfrac{\delta_1}{2k'}\cdot\sin\dfrac{\pi}{2k'}}=\text{const.}$$

for all points of the cubic.

Evidently other relations involving angles—and for (B) of page 35 containing other constants—could be derived in a similar manner, but they would not be simple, and will not be deduced here.

I wish next to derive two or three properties of a cubic from certain non-Euclidian properties of a conic. Such properties of a conic are taken from Professor Story's paper before mentioned. Other properties might be derived in a similar manner.

The relation holds for the conic that the product of the sines of the N. E. D. from any point of the conic to the focal lines of either pair is constant.

That constant is really the parameter of a pencil of conics through four of the absolute points.

If we take as the base-points of the pencil of conics four of the absolute points and as the fixed conics a pair of focal lines and the absolute, the cubic will be generated by this pencil of conics and a pencil of rays. This pencil of conics and the cubic will have the same set of focal lines. The cubic will have a point on each conic of the pencil, hence the product of the sines of the N. E. D., from any point of the cubic to its focal lines of either pair, varies as the parameter of the pencil of conics of which the pair of focal lines is one fixed conic and the absolute the other.

If k_1 and k_2 are the parameters for two different pairs of focal lines, we may write:

The product of the sines of N. E. D. from any point of a cubic to the focal lines of any pair equals $\frac{k_1}{k_2}$ *times the corresponding product for any other pair.*

Next, consider the tangent to a cubic. It will be tangent to one of the conics of the pencil through four of the absolute points; for the pencil contains all the conics through the four points, and a conic through four points and tangent to a given line is possible.

As the tangent to the cubic varies, the conic will vary, but its focal lines and directors, which are also focal lines and directors for the cubic, will not vary. Considering, then, two different focal lines and the directors of those lines, we may, from a property of the conic, write:

The ratio of the sines of the N. E. D. from any tangent to a cubic to any pair of focal lines equals $\frac{k_1}{k_2}$ *times the ratio of the sines of the N. E. D. to the directors of those focal lines.*

Consider any absolute tangent to a cubic, and the polar conic, with respect to the cubic, of the point of contact. This conic will pass through the point of contact and have the same absolute tangent; hence, from a property of a conic, we have:

Any point of contact of the cubic with an absolute tangent is equidistant from the intersections with those tangents of any pair of focal lines of the polar conic of the point of contact. Also, the absolute tangent makes equal angles with the lines joining the point of contact to either pair of foci of the polar conic of the point of contact.

This brings us to the consideration of first and second polars which I reserve for a future paper.